Check out the first newsletter!

4.9 Antiderivatives (continued)

Example:

$$f''(x) = \frac{3}{\sqrt{x}}$$

$$f(1) = 0, f(4) = 1$$

Find $f(x)$.

$f''(x) = 3x^{-1/2}$

$\boxed{f'(x) = 3 \frac{1}{1/2} x^{1/2} + C = 6x^{1/2} + C}$

$f(x) = 6 \frac{2}{3} x^{3/2} + Cx + D$

$\boxed{f(x) = 4x^{3/2} + Cx + D}$

INITIAL CONDITIONS

$f(1) = 0 \Rightarrow 4 + C + D = 0 \Rightarrow C + D = -4$

$D = -4 - C$

$f(4) = 1 \Rightarrow 4(4)^{3/2} + C(4) + D = 1$

$\Rightarrow 32 + 4C + D = 1$

$\Rightarrow 4C + D = -31$

$4C + (-4 - C) = -31$

$3C - 4 = -31$

$3C = -27$

$\boxed{C = -9}$

$\boxed{D = -4 - C = -4 - (-9) = 5}$

$\boxed{f(x) = 4x^{3/2} - 9x + 5}$

CHECK!!!

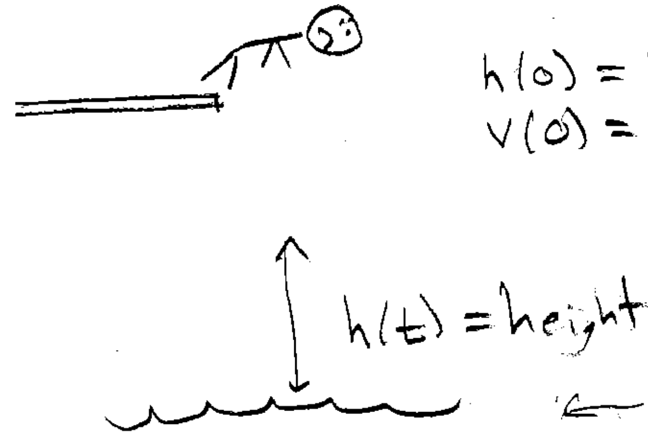
$f(1) = 4 - 9 + 5 = 0$ ✓

$f(4) = 4(4)^{3/2} - 9(4) + 5 = 32 - 36 + 5 = 1$ ✓

$f'(x) = 6x^{1/2} - 9$

$f''(x) = 3x^{-1/2}$ ✓

Example:
Ron *steps off* the 10 meter high dive at his local pool. Find a formula for his height above the water.
(Assume his acceleration is a constant 9.8 m/s^2 downward)



$h(0) = 10$
$v(0) = 0$

$a(t) = -9.8$

$h''(t) = -9.8$, $\quad h(0) = 10$, $h'(0) = 0$

$\Rightarrow h'(t) = -9.8t + C$

$h(t) = -4.9t^2 + Ct + D$

INITIAL CONDITIONS:

$h'(0) = 0 \Rightarrow -9.8(0) + C = 0 \Rightarrow C = 0$

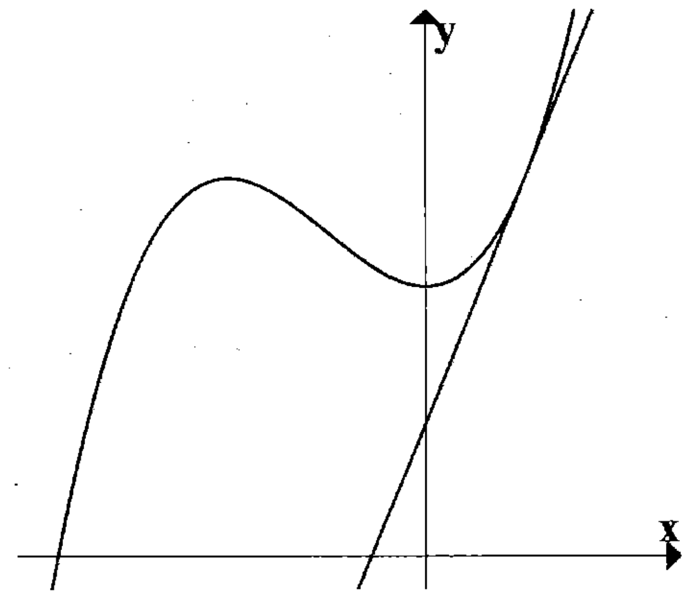
$h(0) = 10 \Rightarrow -4.9(0)^2 + C(0) + D = 10 \Rightarrow \boxed{D = 10}$

$\boxed{h(t) = -4.9t^2 + 10}$ CHECK!

5.1 Defining Area (Riemann sums)

Calculus is based on limiting processes that "approach" the exact answer to a rate question.

Calc. I Visual:

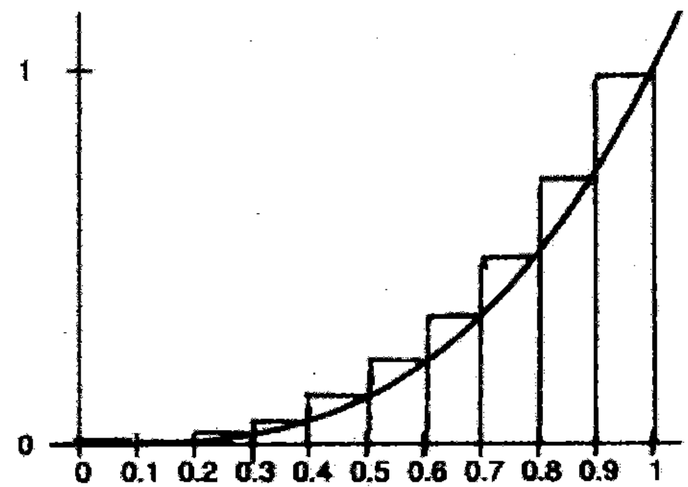


In Calculus I, you defined

f'(x) = `slope of the tangent at *x*'

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Calc. II Visual:



$R_{10} = 0.3025$

In Calculus II, we will see that antiderivatives are related to the area `under' a graph

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

Riemann sums set up:

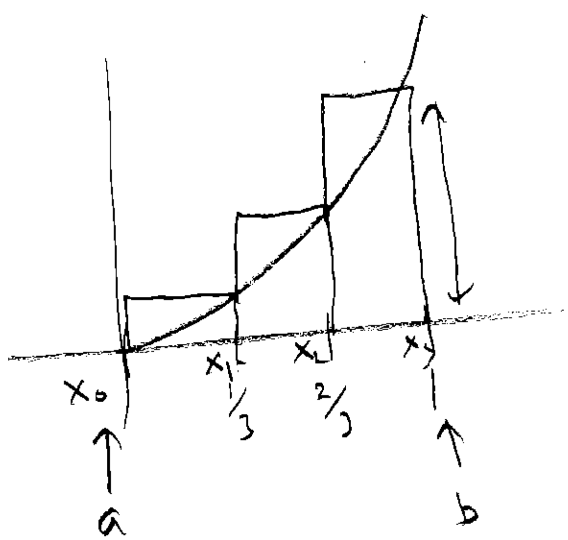
We are going to build a procedure to get better and better approximations of the area "under" $f(x)$.

1. Break into n equal subintervals.

$$\Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x$$

2. Draw n rectangles; use function.
   Area of each rectangle =
   (height)(width) $= f(x_i^*)\Delta x$

3. Add up rectangle areas.



Example:

Approximate the area under $f(x) = x^3$ from x = 0 to x = 1 using n = 3 subdivisions and *right-endpoints* to find the height.

$a = 0$

$b = 1$

$n = 3$

$f(x) = x^3$

$\boxed{\text{I}}$ $\Delta x = \frac{b-a}{n} = \frac{1-0}{3} = 1/3$

$x_0 = a = 0$

$x_1 = a + \Delta x = 0 + 1/3 = 1/3$

$x_2 = a + 2\Delta x = 0 + 2 \cdot 1/3 = 2/3$

$x_3 = a + 3\Delta x = 0 + 3 \cdot 1/3 = 1$

$\boxed{\text{II}}$

$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$

$f\left(\frac{1}{3}\right)\frac{1}{3} + f\left(\frac{2}{3}\right) 1/3 + f(1)\Delta x$

$\left(\frac{1}{3}\right)^3 \frac{1}{3} + \left(\frac{2}{3}\right)^3 \frac{1}{3} + (1)^3 \frac{1}{3}$

$\approx 0.493827 = R_3$

OVERESTIMATE!

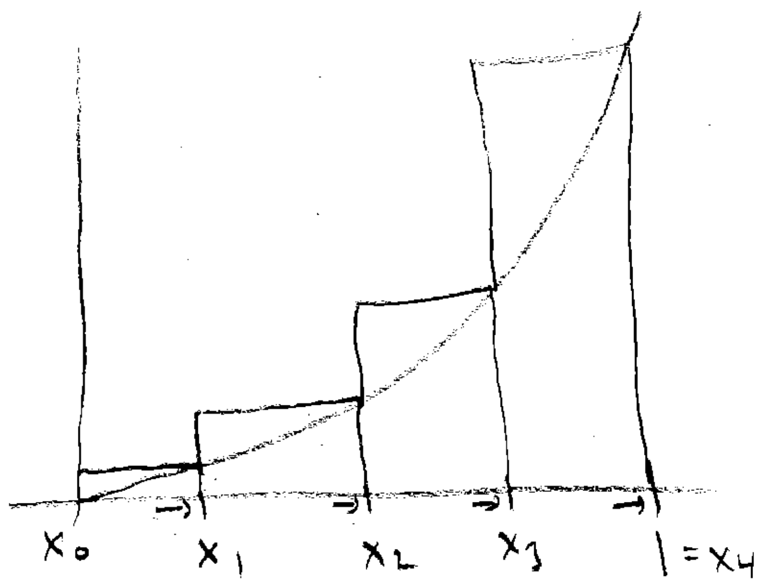
Closing Wed: HW_1A, 1B, 1C

Entry Task (You do): Approx. the area under $f(x) = x^3$ from x = 0 to x = 1 using n = 4 and *right-endpoints*.

Step 1: $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

Step 2:

$$x_0 = a = 0$$
$$x_1 = a + \Delta x = 0 + 1/4$$
$$x_2 = a + 2\Delta x = 0 + 2(1/4)$$
$$x_3 = a + 3\Delta x = 0 + 3(1/4)$$
$$x_4 = a + 4\Delta x = 0 + 4(1/4)$$



Step 3: Plug in right-endpoints to function to get rect. heights, then add up areas (height times width).

$$Area \approx \sum_{i=1}^{4} f(x_i)\Delta x =$$

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$(\ )^3 \frac{1}{4} + (\ )^3 \frac{1}{4} + (\ )^3 \frac{1}{4} + (\ )^3 \frac{1}{4}$$

right-endpoints

$= 0.390625$

$$\left(\frac{1}{4}\right)^3 \frac{1}{4} + \left(\frac{2}{4}\right)^3 \frac{1}{4} + \left(\frac{3}{4}\right)^3 \frac{1}{4} + \left(\frac{4}{4}\right)^3 \frac{1}{4}$$

PATTERN $\boxed{\sum_{i=1}^{4} \left(\frac{i}{4}\right)^3 \frac{1}{4}} = \frac{1}{4^4} \sum_{i=1}^{4} i^3$

ASIDE

did this example again with 100 subdivisions, then 1000, then 10000. Here is a summary of my findings:

n	R_n	L_n
4	0.390625	0.140625
5	0.36	0.16
10	0.3025	0.2025
100	0.255025	0.245025
1000	0.25050025	0.24950025
10000	0.2499500025	0.2500500025

Pattern:

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}, \quad x_i = 0 + i\,\frac{1}{n} = \frac{i}{n}$$

Adding up the area of each rectangle

$$\text{Sum} = \sum_{i=1}^{n} x_i^3 \Delta x = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

$$\text{Area} = 0.25 = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n^4} \sum_{i=1}^{n} i^3 \right)$$

ASIDE

Example: Approximate the area under $f(x) = 1 + x^2$ from $x = 2$ to $x = 3$ using Riemann sums with $n = 4$ and right endpoints.

$$\Delta x = \frac{3-2}{4} = \frac{1}{4}$$

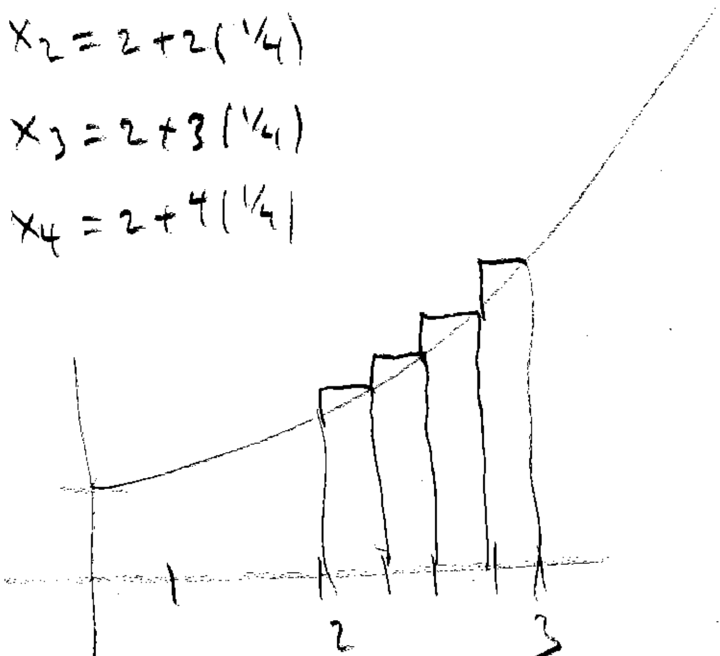
$x_0 = 2$

$x_1 = 2 + \frac{1}{4}$

$x_2 = 2 + 2(1/4)$

$x_3 = 2 + 3(1/4)$

$x_4 = 2 + 4(1/4)$



$$(1+(2.25)^2)\frac{1}{4} + (1+(2.5)^2)\frac{1}{4} + (1+(2.75)^2)\frac{1}{4} + (1+(3)^2)\frac{1}{4} = 7.96875$$

What is the general pattern in terms of *n*?

$$\Delta x = \frac{3-2}{n} = \frac{1}{n}$$

$$x_i = 2 + i\frac{1}{n} = 2 + \frac{i}{n}$$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} (1 + x_i^2)\Delta x$$

$$= \sum_{i=1}^{n} \left(1 + (\underbrace{2}_{a} + \tfrac{i}{n})^2\right) \underbrace{\frac{1}{n}}_{b-a}$$

Another Example:

Using sigma notation, write down the general Riemann sum definition of the area from x = 5 to x = 7 under

$$f(x) = 3x + \sqrt{x}$$

$$\Delta x = \frac{b-a}{n} = \frac{7-5}{n} = \frac{2}{n}$$

$$x_i = a + i\,\Delta x = 5 + i\left(\frac{2}{n}\right) = 5 + \frac{2i}{n}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(3x_i + \sqrt{x_i}\right)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(3\left(5+\frac{2i}{n}\right) + \sqrt{5+\frac{2i}{n}}\right)\frac{2}{n}$$

$a = 5$ $\qquad$ $b - a = 2$

Velocity/Distance & Reimann Sums

When velocity is a ***constant***:

Distance = Velocity · Time

Example:

You are accelerating in a car. You get the following measurements:

t (sec)	0	0.5	1.0	1.5	2.0
v(t) (ft/s)	0	6.2	10.8	14.9	18.1

Estimate the distance traveled by the car traveled from 0 to 2 seconds.

HAVE TO BREAK IT UP!!

	LOW ESTIMATE	HI ESTIMATE
0 to 0.5	$0\ \frac{ft}{sec}$ 0.5 sec = 0 ft	6.2 ft/sec 0.5 sec = 3.1 ft
0.5 to 1	6.2 ft/sec 0.5 sec = 3.1 ft	10.8 ft/sec 0.5 sec = 5.4 ft
1 to 1.5	10.8 ft/sec 0.5 sec = 5.4 ft	14.9 ft/sec 0.5 sec = 7.45 ft
1.5 to 2	14.9 ft/sec 0.5 sec = 7.45 ft	18.1 ft/sec 0.5 sec = 9.05 ft
TOTAL =	15.95 ft	25 ft

ASIDE: UNITS $= \frac{ft}{sec} \cdot sec = ft$

"HEIGHT" UNITS "WIDTH" UNITS

5.2 The Definite Integral

Def'n:
We define the **definite integral of f(x) from x = a to x = b** by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \,, \quad \leftarrow$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

NOTES:

" $\int$ " = integral sign

a, b = bounds or limits of integration

$\int_a^b f(x)dx$ = "ADD UP" $f(x_i)\Delta x$ ACROSS THE INTERVAL

= A NUMBER.

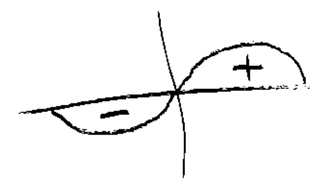
Ex)

$\int_0^3 7dx = 21$

$\int_0^3 -5dx = -15$

"SIGNED" or "NET" AREA

$\int_{-\pi}^{\pi} \sin(x)dx = 0$



Basic Integral Rules:

1. $\int_a^b c\, dx = (b-a)c$

2. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

3. $\int_a^b cf(x)\, dx = c\int_a^b f(x)dx$

and

$\int_a^b f(x) + g(x)\, dx$

$= \int_a^b f(x)dx + \int_a^b g(x)\, dx$

4. $\int_b^a f(x)dx = -\int_a^b f(x)dx$

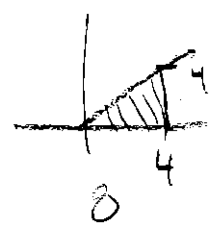
Examples:

1. $\int_4^{10} 5\, dx = 5(10-4) = 30$

2. $\int_0^3 x^2 dx + \int_3^7 x^2 dx = \int_0^7 x^2 dx$

3. $\int_0^4 5x + 3\, dx = \int_0^4 5x\, dx + \int_0^4 3\, dx$

$= 5\int_0^4 x\, dx + \int_0^4 3\, dx$



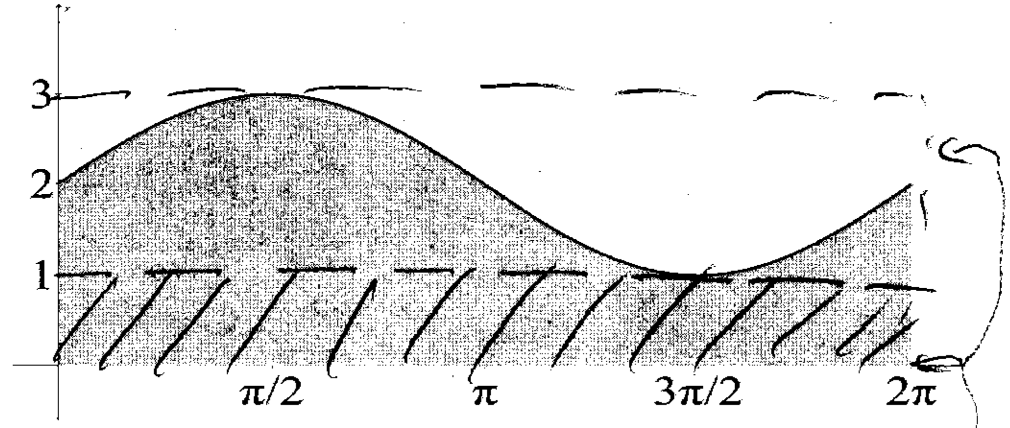
$= 5 \cdot 8 + 12$

$= \boxed{52}$

4. $\int_3^1 x^3 dx = -\int_1^3 x^3\, dx$

Note on quick bounds (HW_1C: 9,10)

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$



Example: Consider the area under

f(x) = sin(x) + 2

on the interval *x = 0* to *x = 2π.*

(a) What is the max of f(x)? (label M)

(b) What is the min of f(x)? (label m)

(c) Draw **one** rectangle that contains all the shaded area? What can you conclude?

(d) Draw **one** rectangle that is completely inside the shaded area? Conclusion?

$$\overset{m}{1} \le \sin(x) + 2 \le \overset{M}{3}$$

SO $\int_0^{2\pi} \sin(x) + 2\,dx$

MUST BE BETWEEN THE AREA OF THESE TWO RECTANGLES

$$\underbrace{1 \cdot 2\pi}_{2\pi} \le \int_0^{2\pi} \sin(x) + 2\,dx \le \underbrace{3 \cdot 2\pi}_{6\pi}$$